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SOLUTION OF SINGULAR RICCATI DIFFERENTIAL EQUATIONS USING THE REPRODUCING KERNEL HILBERT SPACE METHOD

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ABSTRACT

This paper deals with the approximation the solution of singular Riccati differential equations using the reproducing kernel Hilbert space scheme. The exact solution u(r) is represented in the form of series in the spacew_2^2 [0,1]. In the mean time, the n-term approximate solution u(r) obtained and is proved to converge to the exact solution u(r). Some numerical examples have also been studied to demonstrate the accuracy of the present method. Numerical experiments are performed to confirm our theoretic findings.

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KEYWORDS AND PHRASES: Singular Riccati differential equation, Reproducing kernel Hilbert space scheme, exact solution.

1. INTRODUCTION:

In this work, we consider the following quadratic Riccati differential equation with singularity in reproducing kernel space

$$P(r)^{u}{}'(r) = Q(r)u^2(r) + R(r)u(r) + S(r),\ u(0) = \xi, \qquad 0 \le r \le 1, \tag{1}$$

where P(r), Q(r), R(r) and S(r) coefficients are continuous real functions, perhaps P(0) = 0 or P(1) = 0 and $u(r) \in w_2^2[0,1]$

Many physical phenomena such as theory of thermal explosions, studies of electro-hydrodynamics, nuclear physics, gas dynamics, chemical reaction, studies of atomic structures and atomic calculations can be modeled by singular differential equations. Due to this reason, in the literature there exist numerous methods dedicated to the singular problems, for example see [1–3].

The theory of reproducing kernels, firstly has been presented based on the S. Zaremba works in [4] for solving boundary value problems with harmonic and bi-harmonic functions. In recent years, reproducing kernel methods emerged as a powerful framework in numerical analysis, differential, integral equations, probability and statistics [5–7].

Furthermore, extensive works have been proposed and discussed based on their theory for the numerical solutions of singular problems.

For example, using operator transformation considered the solution of Cauchy singular integral equation by reproducing kernel Hilbert space (RKHS) scheme in [8]. Also, in [9] has been used from RKHS method to approximating the solution of the second kind weakly singular Volterra integral equation on graded mesh.

In [10–17] have been solved various classes of singularly boundary value problems. In [18–20] the authors has been implemented the RKHS method for solving singularly perturbed differential-difference equations.

A brief outline of this paper is as follows:
In section 2, RKHS is introduced. In section 3, the analytical solution for (1) in the space $w_2^2[0,1]$ is introduced. Also, an iterative method to solve (1) numerically in the space $w_2^2[0,1]$ is described. In section 4 the numerical examples are discussed.
Then we end with a brief conclusion in section 5.

2. REPRODUCING KERNEL SPACES:

In order to solution of (1), we construct a reproducing kernel functions.

Definition 2.1. ([21]) Let E be a nonempty abstract set and C be the set of complex numbers. A function K: E × E → C is a reproducing kernel of the Hilbert space H if

- for each t ∈ E, K(.,t) ∈ H,
- For each t ∈ E and ψ ∈ H, ⟨ψ (.), K (., t)⟩ = ψ(t).

In Definition 2.1, second condition called the reproducing property, the value of the function ψ at the point t is reproducing by the inner product of $\psi(t)$ with $K(.,t)$.

Remark 2.1. A Hilbert space H of functions on a set E is called a RKHS if there exists a reproducing kernel K of H. That is, a Hilbert space which possesses a reproducing kernel is called the RKHS.

Definition 2.2. The inner product space $w_2^2[0,1]$ is defined as $w_2^2[0,1] = \{x(t)|x(t),x'(t)$, is absolutely continuous, $x(0) = \alpha$ and $x''(t) \in L^2[0,1]\}$.

The inner product $w_2^2[0,1]$ is defined by

$$< x\,(t), y(t) >\ w_2^2[0,1] = x(0)y(0) + x(1)y(1) + \int_0^1 x''\,(t)y''\,(t)dt, \qquad (2)$$

and the norm $\|x\|w_2^2[0,1]$ *is denoted by* $\|x\|w_2^2[0,1] = \sqrt{< x, x >}w_2^2[0,1]$, *where* $x, y \in w_2^2[0,1]$

Theorem 2.1. ([22]) The space $w_2^2[0,1]$ is a reproducing kernel space.

That is, for any $x(t) \in w_2^2[0,1]$ and each fixed $t \in [0,1]$ there exists $R_z(t) \in$

$w_2^2[0,1]$, such that $< x(t), R_z(t) >\ w_2^2[0,1] = x(z)$. The reproducing kernel $R_z(t)$ can be denoted by

$$R_z(t) = \begin{cases} \frac{t}{6}\left((z-1)t^2 + z(z^2 - 3z + 8)\right), t \le z, \\ \frac{z}{6}\left(z^2(t-1) + t(t^2 - 3t + 8)\right), t > z. \end{cases} \qquad (3)$$

Definition 2.3. *The inner product space* $w_2^2[0,1]$ *is defined as* $w_2^2[0,1] = \{x(t)|x(t)$, *is absolutely continuous,* $x(0) = \alpha$ *and* $x'(t) \in L^2[0,1]\}$. *The inner product* $w_2^2[0,1]$ *is defined by*

$$< x\,(t), y(t) >\ w_2^1[0,1] = x(0)y(0) + \int_0^1 x'\,(t)y'\,(t)dt, \qquad (4)$$

and the norm $\|x\|w_2^1[0,1]$ is denoted by$\|x\|w_2^1[0,1] = \sqrt{< x, x >}w_2^1[0,1]$, where $x, y \in w_2^1[0,1]$.

In [5] proved that $w_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$\bar{R}_z(t) = \begin{cases} 1 + t, & t \le z, \\ 1 + z, & t > z, \end{cases} \qquad (5)$$

3. THE ANALYTICAL SOLUTION OF (1)

In this section, the solution of (1) is given in the reproducing kernel space $w_2^1[0,1]$

In (1), it is clear that $\mathcal{L}: w_2^2[0,1] \rightarrow w_2^1[0,1]$ is a bounded linear operator.

Put $\chi_s(z) = \overline{Rz_s}(z)$ and $\phi_s(z) = L{*}\chi_s(z)$, where L∗ is the adjoint operator of L. The orthonormal system $\{\overline{\emptyset}_s(z)\}_{s=1}^{\infty}$ of $w_2^2[0,1]$ can be derived from GramSchmidt orthogonalization process of $\{\emptyset_s(z)\}_{s=1}^{\infty}$

$$\overline{\emptyset}_s(z) = \sum_{i=1}^{s} \propto_{si}\, \emptyset_i\,(z), \quad (\propto_{ss} > 0, s = 1,2,\dots)$$

Theorem 3.1. ([22])Let $\{z_s\}_{s=1}^{\infty}$ is dense on [0,1], then $\{\overline{\emptyset}_s(z)\}_{s=1}^{\infty}$ is the complete system of $w_2^2[0,1]$ and $\phi_s(z) = L_tR_z(t)|_{t=z_s}$.

Theorem 3.2. If $\{z_s\}_{s=1}^{\infty}$ is dense on [0,1] and the solution of (1) is unique, then the solution of (1) satisfies the form

$$v(z) = \sum_{s=1}^{\infty} \sum_{i=1}^{s} \propto_{si} f\left(z_i, v(z_i)\right)\overline{\emptyset}_s(z) \qquad (6)$$

Proof. From Theorem 3.1, one can easily see that $\{\overline{\emptyset}_s(z)\}_{s=1}^{\infty}$ is the complete orthonormal basis of $w_2^2[0,1]$. Remark that from reproducing property $\langle v(z), \chi_s(z)\rangle = v(z_s)$ for each $v(z) \in w_2^2[0,1]$, so we have

$$v(z) = \sum_{s=1}^{\infty} < v(z), \overline{\emptyset}_s(z) > w_2^2[0,1]\overline{\emptyset}_s(z)$$

$$= \sum_{s=1}^{\infty}\sum_{i=1}^{s} \propto si < v(z), \mathcal{L}_{xs}^*(z) > w_2^2[0,1]\overline{\emptyset}_s(z)$$

$$= \sum_{s=1}^{\infty}\sum_{i=1}^{s} \propto si < \mathcal{L}v(z), x_s(z) > w_2^2[0,1]\overline{\emptyset}_s(z)$$

$$= \sum_{s=1}^{\infty}\sum_{i=1}^{s} \propto si < f(z,, v(z)), x_s(z) > w_2^2[0,1]\overline{\emptyset}_s(z)$$

$$= \sum_{s=1}^{\infty}\sum_{i=1}^{s} \propto sif(z_i, v(z_i)), \overline{\emptyset}_s(z),$$

and the proof of the theorem is complete.

3.3 The implementation scheme:

Equation (6) can be denoted by

$$v(z) = \sum_{s=1}^{\infty} A_s\overline{\emptyset}_s(z)$$

,

Where $A_s = \sum_{i=1}^{s} \propto_{si} f(z_i, v(z_i)$. Let $z_1 = 0$, it follows that $f(z_1, v(z_1))$ is known. Considering the numerical computation, we put $v_0(z_1) = v(z_1)$ and define the n-term approximation to $v(z)$ by

$$v_n(z) = \sum_{s=1}^{\infty} B_s\overline{\emptyset}_s(z) \qquad (7)$$

,

where the coefficients Bs and $\phi_s(z)$ are given as

$$B_1 = \propto_{11} \bar{f}(z_1, v_0(z_1)),$$
$$v_1(z) = B_1\overline{\emptyset}_1(z),$$

$$B_1 = \sum_{i=1}^{2} \propto_{2i} f(z_i, v_{i-1}(z_i))$$

$$v_1(z) = \sum_{s=1}^{2} B_s\overline{\emptyset}_s(z),$$

...

$$B_{n-1} = \sum_{i=n-1}^{2} \propto_{2i} f(z_i, v_{i-1}(z_i))$$

$$v_n(z) = \sum_{s=1}^{n} B_s\overline{\emptyset}_s(z).$$

In the iterative process of (7), we can guarantee that the approximation $v_n(z)$ satisfies the initial condition $u(0) = \xi$. On the other hand, the approximate solution $v_n^N(z)$ can be obtained by taking finitely terms in the series of $v_n(z)$ as follow,

$$v_n^N(z) = \sum_{s=1}^{N}\sum_{i=1}^{s} \propto_{si} f(z_i, v_{n-1}(z_i))\overline{\emptyset}_s(z) \qquad (8)$$

4. NUMERICAL SIMULATIONS

In this section, the scheme in the paper will be applied to four numerical examples. All of the computations have been performed by using the Matlab R2010a. Results of each example are compared with exact solution.

Problem 4.1. *In equation* (1), *if* P(r) = r, Q(r) = 1, R(r) = $-\sqrt{r}$ *and* S(r) = -1 *then true solution is*

$$u(r) = \frac{2\sqrt{r-3}}{-2_r + 4\sqrt{r-3}}$$

$$u(0) = 1, r_s = \frac{s-1}{N-1}, s = 1,2, \dots, N$$

RKHS method, taking with the reproducing kernel function Rz(t) on [0,1], the numerical solution $u_n^N(r)$ computed by (8). The numerical results at some selected gird points for N = 51 and n = 7 are given in Table 1.

Table 1: Numerical results for problem 4.1.

r	True solution	Approximate solution	Absolute error	Relative error
0.1	1.223480959455358	1.223480953436179	6.0192×10^{-9}	4.9197×10^{-9}
0.2	1.306879269928204	1.306879257935626	1.1993×10^{-8}	9.1765×10^{-9}
0.3	1.351601504414804	1.351601534632646	3.0218×10^{-8}	2.2357×10^{-9}
0.4	1.366020440416440	1.366020491746745	5.1330×10^{-8}	3.7577×10^{-8}
0.5	1.353553390593274	1.353553470343467	7.9750×10^{-8}	5.8919×10^{-8}
0.6	1.316983583242455	1.316983693532165	1.1029×10^{-7}	8.3744×10^{-8}
0.7	1.259474520209441	1.259474421125751	9.9084×10^{-8}	7.8671×10^{-8}
0.8	1.184736379760737	1.184736316453742	6.3307×10^{-8}	5.3436×10^{-8}
0.9	1.096856471680698	1.096856438463759	3.3217×10^{-8}	3.0284×10^{-8}
1.0	1.000000000000000	1.000000025319537	2.5320×10^{-8}	2.5320×10^{-8}

Problem 4.2. *Consider the following singular equation*

$(1-r)u'(r) = u^2(r) + u(r), \quad u(0) = -\frac{1}{2}, \quad 0 \le r < 1$ *, with true solution*

$$u(r) = \frac{1}{r-2}$$

$$r_s = \frac{s-1}{N-1}, s = 1,2, \dots, N$$

Using our method, taking and gird points N = 51 *and n* = 9, *the numerical results are as given in Figure 1.*

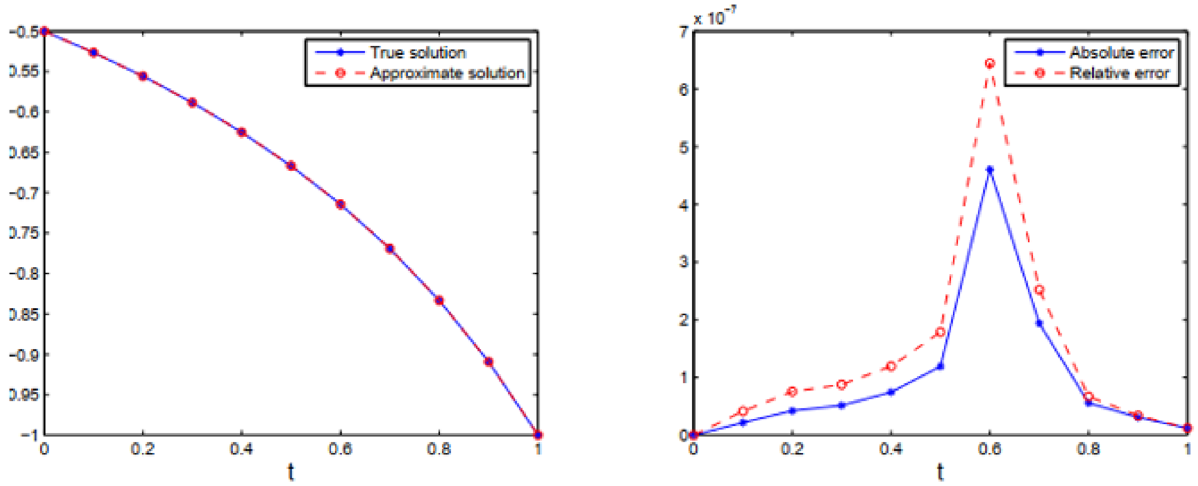


Figure 1: Comparisons of approximate solution with the exact solution (left) and the absolute errors with the relative errors of problem 4.2 (right).

Problem 4.3. *Consider the following singular equation*

$$u'(r) - \sqrt{r}u^2(r) + \frac{u(r)}{\sqrt{r}}, \quad u(0) = 1, \quad 0 < r \leq 1$$

$$u(r) = \frac{2}{1 + 2\left(r - \sqrt{r}\right) + e^{-2\sqrt{r}}}$$

.

by RKHS method, taking $r_s = \frac{s-1}{N-1}, s = 1,2, \dots, N$ *on* $[0,1]$. *The numerical results at some selected gird points for N = 53 and n = 9 are given in Table 2.*

Table 2: Numerical results for problem 4.3.

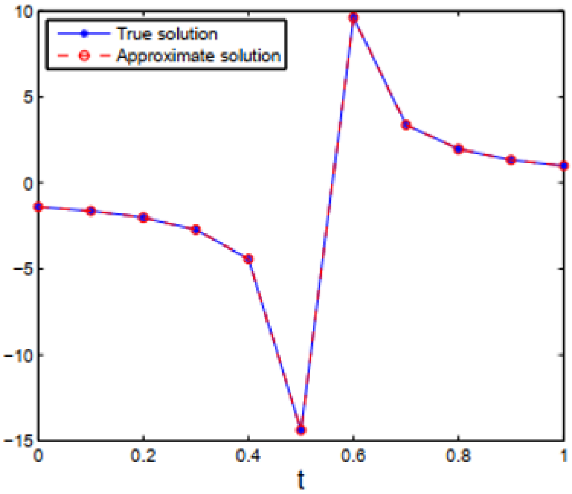
r	True solution	Approximate solution	Absolute error	Relative error
0.1	1.820117633910813	1.820117625479621	8.4312×10^{-9}	4.6322×10^{-9}
0.2	2.187191844632269	2.187191765623859	7.9008×10^{-8}	3.6123×10^{-8}
0.3	2.383944752470398	2.383944725739516	2.6731×10^{-8}	1.1213×10^{-8}
0.4	2.446922176511700	2.446921486935627	6.8958×10^{-7}	2.8181×10^{-7}
0.5	2.412827055573784	2.412826748392754	3.0718×10^{-7}	1.2731×10^{-7}
0.6	2.316890586034218	2.316890058361949	5.2767×10^{-7}	2.2775×10^{-7}
0.7	2.187458502978635	2.187457726387531	7.7659×10^{-7}	3.5502×10^{-7}
0.8	2.044368584151155	2.044368509754165	7.4397×10^{-8}	3.6391×10^{-8}
0.9	1.900063469836569	1.900063425736172	4.4100×10^{-8}	2.3210×10^{-8}
1.0	1.761594155955765	1.761594138936148	1.7020×10^{-8}	9.6615×10^{-9}

Problem 4.4. Consider the following singular equation

$$(1 - e^{1-r})u'(r) = u^2(r), \quad u(0) = \frac{1}{2-e}, \quad 0 \leq r < 1$$, *with true solution*

$$u(r) = \frac{1}{\mathrm{e}r + 1 - \mathrm{e}}$$

Using our method, take $r_s = \frac{s-1}{N-1}, s = 1,2, \dots, N$ $N = 35$. *and n = 7. The numerical results are given in Figure 2.*



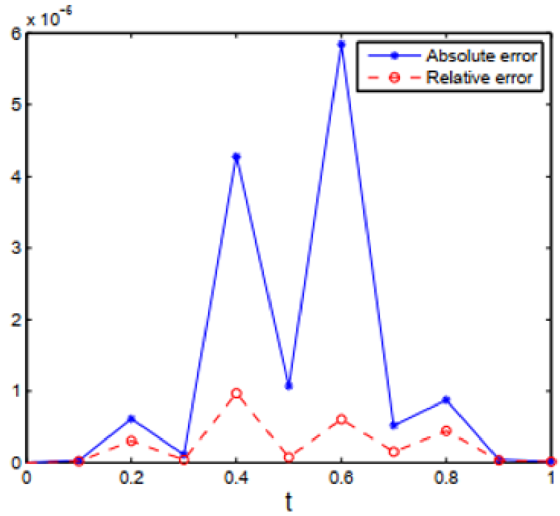


Figure 2: Comparisons of approximate solution with the exact solution (left) and the absolute errors with the relative errors of problem 4.4 (right).

5. CONCLUSIONS:

In this paper, the reproducing kernel Hilbert space scheme was implemented for solving a singular Riccati differential equations. This confirms the validity of the present method and it is efficient, accurate and reliable for singular Riccati differential equations.

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